Continuous and Differentiable Functions Exploration using TI-Nspire CAS
Mathematical Methods CAS Unit 3

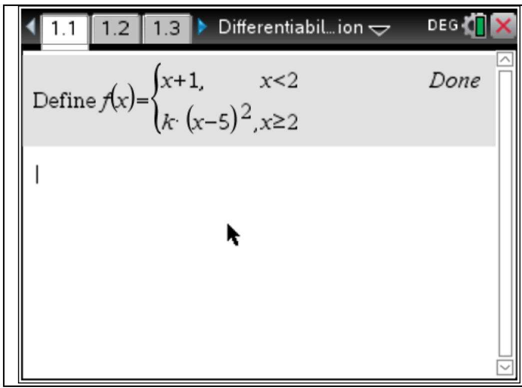
Objective: Given a hybrid function, make the function continuous at the boundary between the two branches. Then make the function differentiable at this point.

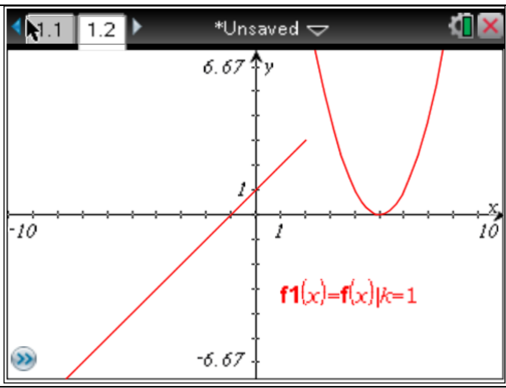
Let *f* be the function defined by

$$f(x)=\begin{cases} x+1, & x<2 \\ k(x-5)^2, & x\geq 2 \end{cases}$$

where k is a constant.

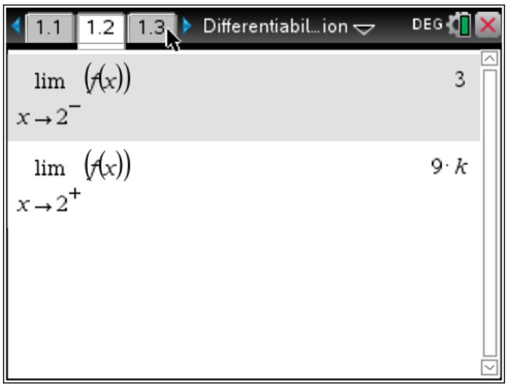
1. Sketch the graph of *f* for $k=1$.



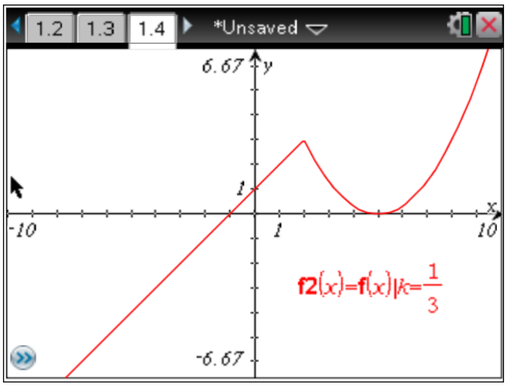


2. Function is discontinuous at $x=2$. Explain what it means for the function to be discontinuous.

3. Find $\lim_{x\to 2^-} f(x)$ and $\lim_{x\to 2^+} f(x)$. The second limit will be in terms of k. What must be true of these two limits for *f* to be continuous at $x=2$?

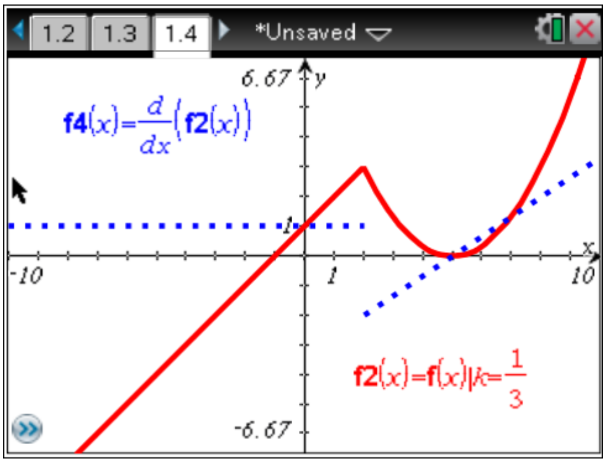


4. Find the value of k that makes *f* continuous at $x=2$. Sketch the graph of *f* for this value of k.



5. The graph in part 4 has a cusp at $x = 2$. Cusp comes from the Latin *cuspis*, meaning a point or a pointed end. Why is it appropriate to use the word cusp in this context?

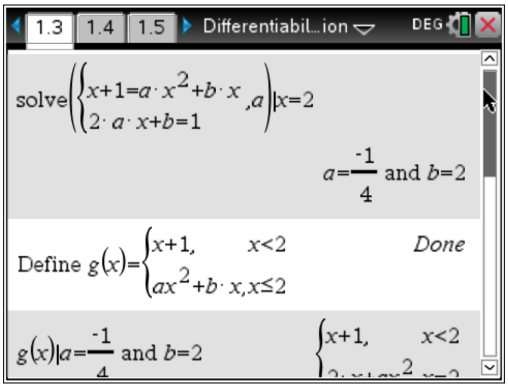
6. Suppose someone asks, 'Is *f(x)* increasing or decreasing at $x = 2$ with *k* as in part 4?" How would you have to answer that question? What, then, can you conclude about the derivative of a function at a point where the graph has a cusp?
7. Sketch the graph of $f'(x)$ for the value of *k* you found in part 4.



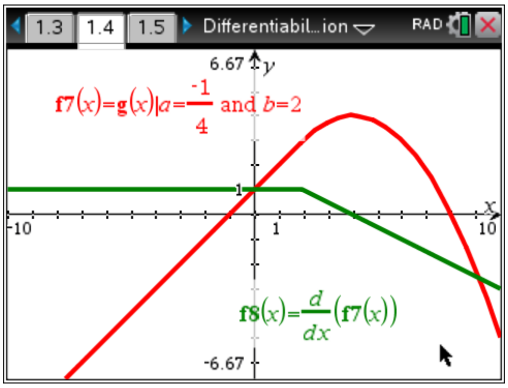
8. **For two graphs to join smoothly, the gradients on both sides have to be equal**. Let's define a new function g:

$$g(x) = \begin{cases} x+1, & x < 2 \\ ax^2 + bx, & x \geq 2 \end{cases}$$

Find the values of *a* and *b* so that both graphs join **smoothly.**



9. Sketch the graph of $g(x)$ and $g'(x)$.



ABSOLUTE VALUE FUNCTION

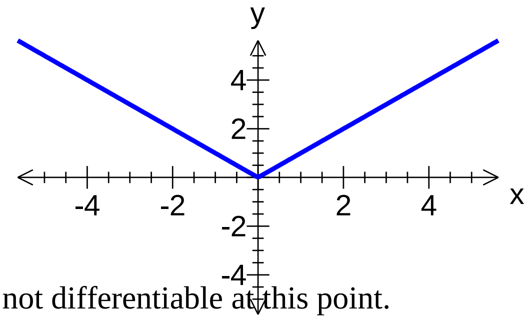
Consider the absolute value function

$$f(x)=|x|=\begin{cases} x, & x\geq 0 \\ -x, & x<0 \end{cases}$$

Is *f(x)* differentiable at $x=0$? Justify your answer.

Solution

Recall the graph of the absolute value function:



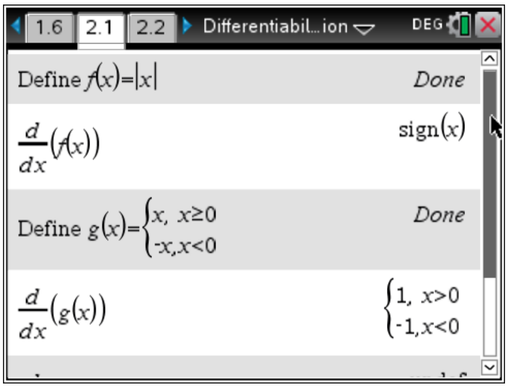
It clearly has a sharp point at $x=0$, so the function is not differentiable at this point.

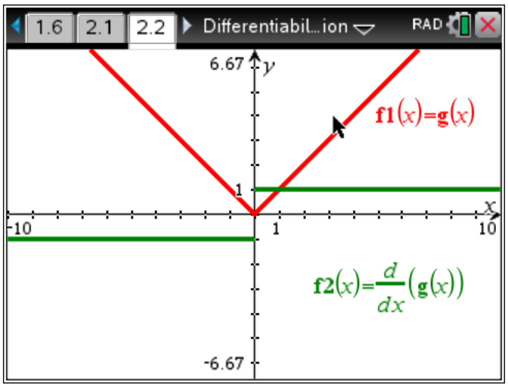
Analytically:

$$f'(x)=\begin{cases} 1, & x>0 \\ -1, & x<0 \end{cases}$$

And $f'(x)$ is undefined at $x=0$.

Check on your CAS calculator.





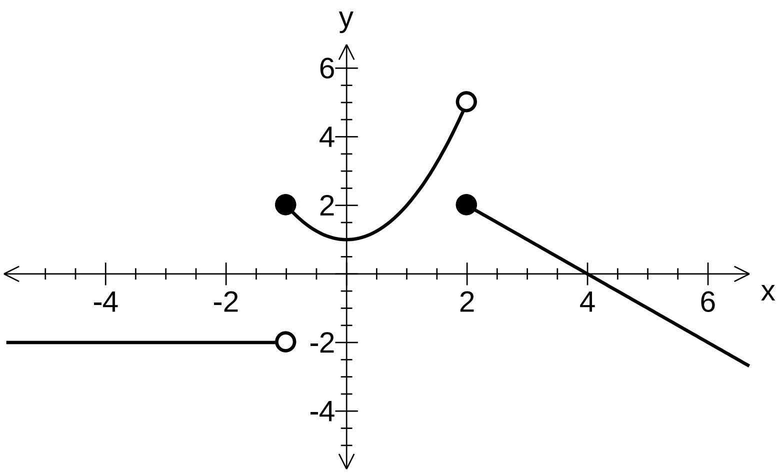
ADDITIONAL QUESTIONS:

Q1. Given that $g(x)=\begin{cases} x^2-6x+8, & 1\le x<4 \\ mx+2, & -5\le x<1 \end{cases}$, determine the value of m for which $g(x)$ is continuous on $[-5, 4)$.

Q2. For your value of m, sketch the graph of g'.

Q3. Find the values of m and n so that $f'(1)$ exists for the hybrid function:
$$f(x)=\begin{cases} mx^2+n, & x\le 1 \\ \dfrac{1}{x}, & x>1 \end{cases}$$

Q4. Sketch the gradient graph for the function drawn below:



Critical points versus stationary points.

If a function f has any relative extrema (maxima or minima), then they occur either at points where $f'(x) = 0$ (stationary points) or at points where function f is not differentiable.

Example: For the function $f(x) = |x - 2| + 3$, there is a critical point, a local minimum at (2, 3). However, it is not a stationary point because *f(x)* is not differentiable at *x* = 2 (sharp point).