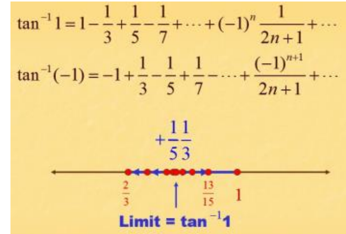
Applications with Maclaurin Power Series



After completing this lesson, you will be able to:

- Differentiate and integrate power series.
- Evaluate limits using Taylor series.

In this lesson you will differentiate and integrate Taylor and Maclaurin series. Further on you will use those series to evaluate limits. It is important to note that differentiation and integration of series is only valid on the interval of convergence of the series.

Let start with looking at the list of Maclaurin series for some of the most important functions, together with a specification of the intervals over which the Maclaurin series converge to those functions. Most of these results have been already derived in the previous lessons.

DIFFERENTIATING

Consider the Maclaurin series for sin *x*:

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} \ldots$$

Then

$$\frac{d}{dx}\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \ldots \right) = 1 - 3\frac{x^2}{3!} + \frac{5x^4}{5!} - \frac{7x^6}{7!} + \ldots$$

$$= 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \ldots = \cos x$$

Here is another example:

$$\frac{d}{dx}\left(e^x\right) = \frac{d}{dx}\left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \ldots \right)$$

$$= 1 + \frac{2x}{2!} + \frac{3x^2}{3!} + \frac{4x^3}{4!} + \ldots.$$

$$= 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \ldots = e^x$$

Interestingly, we have just proved, using Maclaurin series, that derivative of e^x is e^x.

INTEGRATING power series works in the similar way:

$$\int \cos x\,dx = \int \left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \ldots\right) dx = x - \frac{x^3}{3(2!)} + \frac{x^5}{5(4!)} - \frac{x^7}{7(6!)} + \ldots$$

$$= x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} \ldots + c$$

$$= \sin x + c$$

The same idea applies to definite integrals.

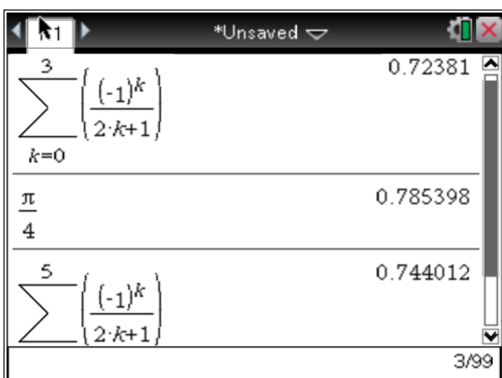
$$\int_0^1 \frac{dx}{1+x^2} = \arctan x\Big]_0^1 = \frac{\pi}{4}$$

But we can express $\frac{1}{1+x^2} = 1 - x^2 + x^4 - x^6 + \ldots$ by substituting $-x^2$ for x in the expansion

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + x^4 + \ldots (-1 < x < 1)$$

$$\therefore \int_0^1 \frac{dx}{1+x^2} = \int_0^1 \left(1 - x^2 + x^4 - x^6 + \ldots\right) dx = \left[x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \ldots\right]_0^1 = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \ldots.$$

This allows us to approximate the value of $\frac{\pi}{4} \approx 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \ldots.$



The convergence of Maclaurin series for arctan(x) is quite slow at the endpoint $x = 1$.

POWER SERIES FOR A DEFINITE INTEGRAL EXPLORATION

Example 1

Approximate the integral $\int_0^1 e^{-x^2}\,dx$ to three-decimal place accuracy by expanding the integrand in a Maclaurin series and integrating term by term.

Solution

$$e^{x} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \dots$$

$$e^{-x^2} = 1 - x^2 + \frac{x^4}{2!} - \frac{x^6}{3!} + \frac{x^8}{4!} + \dots$$

$$\int_0^1 e^{-x^2}\,dx = \int_0^1 \left(1 - x^2 + \frac{x^4}{2!} - \frac{x^6}{3!} + \frac{x^8}{4!} + \dots\right) dx$$

$$= x - \frac{x^3}{3} + \frac{x^5}{5(2!)} - \frac{x^7}{7(3!)} + \frac{x^9}{9(4!)} + \dots$$

$$= 1 - \frac{1}{3} + \frac{1}{5(2!)} - \frac{1}{7(3!)} + \frac{1}{9(4!)} + \dots$$

$$= \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)k!}$$

It is an alternating series and converges $\lim_{k\to\infty} \frac{1}{(2k+1)k!} = 0$

The truncation error is less than the first rejected term. For 3 decimal place accuracy we must choose *n* such that

$$\frac{1}{\left[2(n+1)+1\right](n+1)!} \le 5\times 10^{-4}$$

$$\frac{1}{(2n+3)(n+1)!} \le 0.0005$$

By trial and error $n = 5$

So we must take five terms.

$$\int_0^1 e^{-x^2}\,dx = 1 - \frac{1}{3} + \frac{1}{5(2!)} - \frac{1}{7(3!)} + \frac{1}{9(4!)} \approx 0.747$$

Using GDC: $\int_0^1 e^{-x^2}\,dx = 0.746824$

TRY THESE:

Use Maclaurin series to approximate the integrals to 3 d.p. accuracy:

1. $\int_0^1 \sin\left(x^2\right) dx$

2. $\int_0^{\frac{1}{2}} \arctan\left(2x^2\right) dx$

ANSWER

1. 0.3103 2. 0.081

LIMITS

The limit of an indeterminate form as $x \to a$ can sometimes be found without using L'Hospital's rule by expanding the functions involved in Taylor series about $x = a$ and taking the limit of the series term by term.

Example 2

Use this method to find the following limits:

a) $\lim_{x \to 0} \frac{\sin x}{x}$ b) $\lim_{x \to 0} \frac{\arctan(x) - x}{x^3}$

Solution

a)

$$\lim_{x \to 0} \frac{\sin x}{x} = \lim_{x \to 0} \frac{x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots}{x}$$

$$= \lim_{x \to 0} \left(1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \frac{x^6}{7!} + \dots \right)$$

$$= 1$$

Example 3

Find the Maclaurin series for $\sin^2 x$ using the series for $\cos 2x$. Hence find $\lim_{x \to 0} \frac{\sin^2 x - x^2}{x^4}$.

Solution

We can use a double angle formulae $\sin^2 x = \frac{1 - \cos 2x}{2}$

Maclaurin series for $\cos 2x = 1 - \frac{(2x)^2}{2!} + \frac{(2x)^4}{4!} - \frac{(2x)^6}{6!} + \dots$

$$\sin^2 x = \frac{1-\left(1-\frac{(2x)^2}{2!}+\frac{(2x)^4}{4!}-\frac{(2x)^6}{6!}+\ldots\right)}{2}$$

$$= \frac{\frac{2x^2}{2}-\frac{16x^4}{24}+\frac{64x^6}{720+\ldots}}{2}$$

$$= x^2-\frac{x^4}{3}+\frac{2x^6}{45}+\ldots$$

$$\lim_{x\to 0}\frac{\sin^2 x - x^2}{x^4} = \lim_{x\to 0}\frac{\cancel{x^2}-\frac{x^4}{3}+\frac{2x^6}{45}+\ldots-\cancel{x^2}}{x^4}$$

$$= \lim_{x\to 0}\frac{-\frac{x^4}{3}+\frac{2x^6}{45}+\ldots}{\underset{x\to 0}{x^4}} = \lim_{x\to 0}\left(-\frac{1}{3}+\frac{2x^2}{6}+\ldots\right)$$

$$= -\frac{1}{3}$$

Example 4

a) Using the Maclaurin series for $f(x) = \arctan x$ up to and including the term with x^{13}, show that

$$\int_0^{\frac{1}{\sqrt{3}}} \arctan x \, dx \approx 0.158459.$$

b) What is the absolute value of the error in this approximation?

c) Find the exact value of the integral $\int_0^{\frac{1}{\sqrt{3}}} \arctan x \, dx.$

d) Hence deduce that an approximate value of π is $\pi \approx 3.14159.$

e) To how many decimal places is this approximation accurate?

Solution

a)

$$\int_0^{\frac{1}{\sqrt{3}}} x-\frac{x^3}{3}+\frac{x^5}{5}-\frac{x^7}{7}+\ldots.+\frac{x^{13}}{13}\,dx$$

$$= \left[\frac{x^2}{2}-\frac{x^4}{12}+\frac{x^6}{30}-\frac{x^8}{56}+\frac{x^{10}}{90}-\frac{x^{12}}{132}-\frac{x^{14}}{182}\right]_0^{\frac{1}{\sqrt{3}}}$$

$$\approx 0.158459$$

b) it is a converging alternating series, the error is less than the first unused term which is $-\frac{x^{15}}{210}$.

$$\left|error\right| < 0.000001$$

c)

Using integration by parts:

$$\int_0^{\frac{1}{\sqrt{3}}} \arctan x\, dx = \left[x \arctan x\right]_0^{\frac{1}{\sqrt{3}}} - \int_0^{\frac{1}{\sqrt{3}}} \frac{x}{1+x^2}\, dx$$

$$= \frac{\frac{\pi}{6}}{\sqrt{3}} - \left[\frac{1}{2}\ln\left(1+x^2\right)\right]_0^{\frac{1}{\sqrt{3}}} = \frac{\pi}{6\sqrt{3}} - \frac{1}{2}\ln\left(\frac{4}{3}\right)$$

d)

$$\frac{\pi}{6\sqrt{3}} - \frac{1}{2}\ln\left(\frac{4}{3}\right) \approx 0.158459$$

$$\pi = 6\sqrt{3}\left[0.158459 + \frac{1}{2}\ln\left(\frac{4}{3}\right)\right] = 3.14159$$

e) to 5 decimal places because $\left|error\right| < 0.000001$

Example 5

Find the first 3 terms in the Maclaurin series for $\cos(\sin x)$. Hence find $\lim_{x \to 0} \frac{1-\cos(\sin x)}{x^2}$.

Solution

From

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \ldots$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \ldots$$

$$\cos(\sin x) = 1 - \frac{\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \ldots\right)^2}{2!} + \frac{\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \ldots\right)^4}{4!} - \ldots.$$

$$= 1 - \left(\frac{x^2}{2!} - \frac{2x^4}{2!3!} + \ldots.\right)$$

We only need those terms which will not be zero when $x = 0$ in the limit calculations.

$$\lim_{x\to 0}\frac{1-\cos(\sin x)}{x^2}=\lim_{x\to 0}\frac{1-\left(1-\frac{x^2}{2}+\frac{x^4}{6}-\ldots\right)}{x^2}=\lim_{x\to 0}\frac{\frac{x^2}{2}-\frac{x^4}{6}+\ldots}{x^2}$$

$$=\lim_{x\to 0}\left(\frac{1}{2}-\frac{x^2}{24}+\ldots\right)=\frac{1}{2}$$

Example 6

Use the Maclaurin series for the functions e^x and $\sin x$ to expand $e^{\sin x}$ up to the term in x^4. Hence integrate $\int_0^1 e^{\sin x}\,dx$.

Solution

$$e^x=1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\ldots$$

$$\sin x=x-\frac{x^3}{3!}+\frac{x^5}{5!}-\ldots$$

so that

$$e^{\sin x}=1+\sin x+\frac{\sin^2 x}{2!}+\frac{\sin^3 x}{3!}+\frac{\sin^4 x}{4!}+\ldots$$

$$=1+\left(x-\frac{x^3}{3!}+\frac{x^5}{5!}-\ldots\right)+\frac{\left(x-\frac{x^3}{3!}+\frac{x^5}{5!}-\ldots\right)^2}{2!}+\frac{\left(x-\frac{x^3}{3!}+\frac{x^5}{5!}-\ldots\right)^3}{3!}+\frac{\left(x-\frac{x^3}{3!}+\frac{x^5}{5!}-\ldots\right)^4}{4!}+\ldots$$

$$=1+x-\frac{x^3}{3!}+\ldots+\frac{x^2-\frac{2x^4}{3!}+\ldots}{2!}+\frac{x^3+\ldots}{3!}+\frac{x^4+\ldots}{4!}+\ldots$$

$$=1+x-\frac{x^3}{3!}+\frac{x^2}{2!}-\frac{2x^4}{2!3!}+\frac{x^3}{3!}+\frac{x^4}{4!}+\ldots$$

$$=1+x+\frac{x^2}{2}-\frac{x^4}{8}+\ldots$$

Hence the integral becomes

$$\int_0^1 e^{\sin x}\,dx=\int_0^1 1+x+\frac{x^2}{2}-\frac{x^4}{8}+\ldots dx$$

$$=\left[x+\frac{x^2}{2}+\frac{x^3}{6}-\frac{x^5}{40}\right]_0^1=\frac{394}{240}\approx 1.6417$$

The GDC gives $\int_0^1 e^{\sin x}dx = 1.63187$

Example 7

Use the fact that $\int \frac{1}{1+x^2}dx = \arctan x + c$ and the power series $\frac{1}{1+x^2} = \sum_{k=0}^{\infty}(-1)^k x^{2k} = 1 - x^2 + x^4 - x^6 + \ldots$ to find the Maclaurin series for arctanx.

Solution

$$\frac{1}{1+x^2} = 1 - x^2 + x^4 - x^6 + \ldots$$

$$\arctan x + c = \int 1 - x^2 + x^4 - x^6 + \ldots dx$$

$$\arctan x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \ldots - c$$

The constant of integration can be evaluated by substituting x = 0 and using the condition that arctan 0 = 0.
This gives c = 0.
Therefore:

$$\arctan x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \ldots$$

TRY THIS

a) Write down the Maclaurin series for $f(x) = \frac{\sin x}{x}$

b) Use this series to find an approximate value of the integral $\int_0^1 \frac{\sin x}{x}dx$ by keeping the first three terms in the Maclaurin series.

c) Estimate the error made by evaluating this integral in this way.

d) Confirm your answer by GDC.

SOLUTION

a)

$$\frac{\sin x}{x} = 1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \frac{x^6}{7!} + \ldots$$

b)

$$\int_0^1 \frac{\sin x}{x}dx = \int_0^1 \left(1 - \frac{x^2}{3!} + \frac{x^4}{5!} + \ldots\right)dx$$

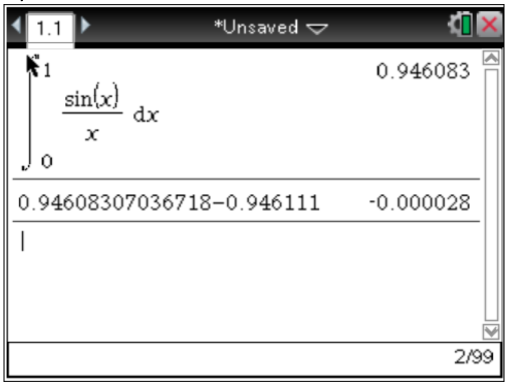
$$\int_0^1 \frac{\sin x}{x}dx = 1 - \frac{1}{3 \cdot 3!} + \frac{1}{5 \cdot 5!} \approx 0.946111$$

c)

$$|error| < |u_4|$$

$$\left| -\frac{1}{7 \cdot 7!} \right| = 0.0000283$$

d)



So the error is less than 0.0000283.

Example 8

How can we proceed to find the Maclaurin series for $f(x) = \frac{1}{1-x^2}$?

ANSWER

We can use the known series $\frac{1}{1-x} = \sum_{k=0}^{\infty} x^k = 1 + x + x^2 + x^3 + \ldots$ and resolve $f(x) = \frac{1}{1-x^2}$ into partial fractions:

$$\frac{1}{1-x^2} = \frac{1}{2}\left(\frac{1}{1-x} + \frac{1}{1+x}\right)$$

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \ldots$$

$$\frac{1}{1+x} = 1 - x + x^2 - x^3 + \ldots$$

Thus

$$\frac{1}{1-x^2} = \frac{1}{2}\left(1 + x + x^2 + x^3 + \ldots + 1 - x + x^2 - x^3 + \ldots\right)$$

$$= 1 + x^2 + x^4 + x^6 + \ldots$$

And is valid for $0 < x < 1$

SOME MORE PRACTICE:

QUESTION

a) Find the first four terms in the Maclaurin series for $\sin^2 x$.

b) Evaluate $\int_0^{\frac{\pi}{6}} \sin^2 x\,dx$ using the first three terms.

c) Estimate the error involved.

d) Find the exact value of $\int_0^{\frac{\pi}{6}} \sin^2 x\,dx$.

e) Confirm the error made using the first three terms.

ANSWER

a) We cannot use substitution here so we have to find the series by finding consecutive derivatives. This results in

$$\sin^2 x = x^2 - \frac{x^4}{3} + \frac{2x^6}{45} - \frac{x^8}{315} + \ldots$$

b)

$$\int_0^{\frac{\pi}{6}} \sin^2 x\,dx = \int_0^{\frac{\pi}{6}} x^2 - \frac{x^4}{3} + \frac{2x^6}{45} - \ldots dx$$

$$\approx 0.0452941$$

c)

$$\left|error\right| < \left| \left[-\frac{x^9}{9 \cdot 315} \right]_0^{\frac{\pi}{6}} \right|$$

$$\left|error\right| < 1.04 \times 10^{-6}$$

d)

$$\int_0^{\frac{\pi}{6}} \sin^2 x\,dx = \frac{1}{2}\int_0^{\frac{\pi}{6}} (1 - \cos 2x)\,dx$$

$$= \frac{1}{2}\left[\frac{\pi}{6} - \frac{\sqrt{3}}{4}\right] \approx 0.045293$$

e) error: 0.045293-0.00000104=-1.033x10^{-6}. So the sign is the same as the sign of the first unused term and it is consistent with the alternating series estimation theorem.