Solutions to Disappearing Wombats

Question 1

$$\frac{dW}{dt} = (0.10 - .06)W = 0.04W$$

$$\int \frac{dW}{W} = 0.04t + c$$

$$\log_e W = 0.04t + c$$

When $t = 0$, $W = 200$, hence $c = \log_e 200$

$$0.04t = \log_e W - \log_e 200 = \log_e \left(\frac{W}{200}\right)$$

$W = 200e^{0.04t}$ $(t \geq 0)$

i. $t = 10$, $W = 298$ (to nearest wombat)
ii. $t = 100$, $W = 10\,920$ (to nearest wombat)

Question 2

The following situations should be explored; cases shown in part **a.** demonstrating each of the possibilities shown in part **b.**

i. $m = n$ (possible for $0.08 \leq m, n \leq 0.09$)
this will give a constant population of 200.

ii. $m > n$ ($m - n$ can be as large as 0.06)
in all such cases, $W = 200\ e^{(m-n)t}$ will give exponential increase without limit, the larger the value of $m - n$, the faster the growth.

iii. $m < n$ (only possible if $m < 0.9$, $n > 0.8$) $m - n$ will be negative, hence population will decay and eventually die out. Lowest value possible is $m - n = -0.01$.

c. Various limitations, but most obviously the impossibility of infinite population. Also discrete rather than continuous variable, birth/death rates assumed independent of *W*, environmental factors ignored and so on. Model may give useful predictions for a limited period of time after arrival.

Question 3

$-kW$ has effect of reducing the overall growth rate as W increases. Eventually, if $k > 0$, growth rate must become negative as $\frac{dW}{dt} = (m-n)W - kW^2$ which is a quadratic with negative W^2 term.
[this could be explained in a number of acceptable ways]

Question 4

$$\frac{dW}{dt} = (m - n - kW)W$$

[students might substitute earlier for *m*, *n*, *k*]

$$\int \frac{dW}{(m-n-kW)W} = \int dt$$

using partial fractions, $\dfrac{1}{(m-n)}\displaystyle\int \left(\frac{1}{W} + \frac{K}{(m-n-kW)}\right) dW = t + c$

$$\frac{1}{(m-n)}(\log_e |W| - \log_e |m-n-kW|) = t + c$$

rearranged, this gives: $W = [(m-n) - kW]Ae^{(m-n)t}$ $\left[W < \dfrac{m-n}{k}\right]$

if $m = 0.10$, $n = 0.06$ and $k = 0.00005$ and assuming $W = 200$ when $t = 0$,

$$A = \frac{200}{0.04 - 0.01} = \frac{20000}{3}$$

rearranging, $W = \dfrac{800e^{0.04t}}{3 + e^{0.04t}}$, $(t \geq 0)$

after 10 years, $W \cong 266$ (to nearest wombat)

after 100 years, $W \cong 758$ (to nearest wombat)

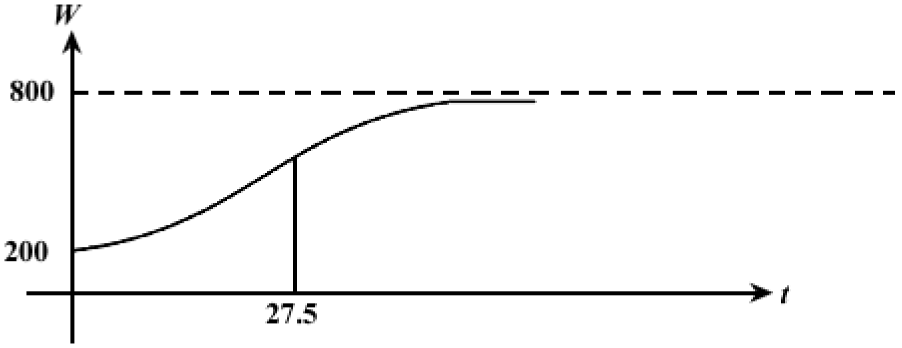
Relatively small difference with Model 1 after 10 years, but extremely significant after 100 years. Notice that $W \to 800$ as t becomes very large, hence there is an upper limit on wombat population.

$$\frac{dW}{dt} = 0.04W - 0.00005W^2$$

hence $\dfrac{d^2W}{dt^2} = (0.04 - 0.0001\,W)\dfrac{dW}{dt}$

max of $\dfrac{dW}{dt}$ occurs when $W = 400$ which gives $t \cong 27.5$

Wombat population increasing most rapidly at this time.
[Students are more likely to find 2nd differential of expression for *W*, though this is more cumbersome. Trial and error also reasonable here.]



Question 5

a., b., c.

Provided $m > n$, the following can be observed:

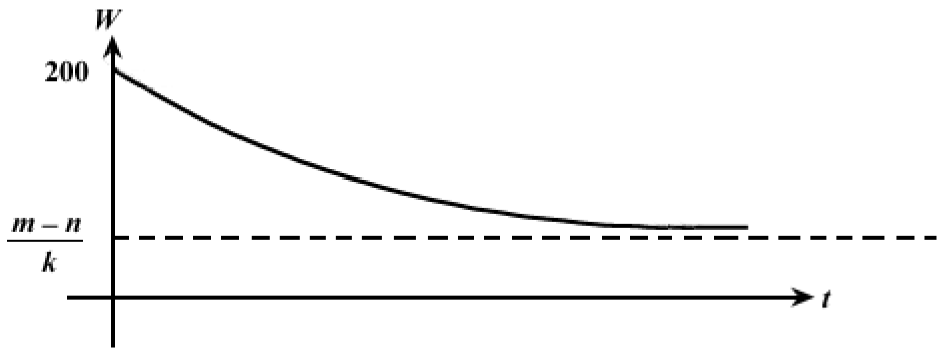
$$W \to \frac{m-n}{k} \qquad \text{as } t \to \text{infinity}$$

thus the graphs will be of three general types, depending on whether $\frac{m-n}{k}$ is less than, greater than, or equal to 200

i. $\frac{m-n}{k} > 200$, essentially like case in Question 4.

ii. $\frac{m-n}{k} = 200$, stable population

iii. $\frac{m-n}{k} < 200$, population will decay asymptotically to $\frac{m-n}{k}$ as shown in graph below.



In case **iii.** care must be taken with the integral to ensure that ($m - n - kw$) is positive. The use of the modulus above takes care of this. However, when rearranging to find *W*, resulting expression is slightly different to Question 4.

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