

Proof by Mathematical Induction.

Mathematicians have very peculiar characteristics. They like proving things or mathematical statements. Two of the most important techniques of mathematical proof are proof by induction and proof by contradiction.



You may wonder what constitutes a proper mathematical proof? It is a complicated question and we need to go deeper into the structure of mathematics to be able to answer this question.

The following Power Point presentation shows the history of mathematics and its structure. It looks at axioms and Euclidean geometry. It explains how mathematics is based on axioms and how we build more and more valid theorems. It also looks at various methods of proofs. Finally it asks those philosophical questions you must have heard in TOK.

Proof

A proof is a careful argument that establishes a new fact or theorem, given certain assumptions or hypotheses. There are various kinds of proof, some of which we mention here.

Proof by deduction

A deductive proof consists of a sequence of statements or sentences each of which is deduced from previous ones or from hypotheses using standard mathematical properties. The final statement may be called a theorem.

Proof by contradiction

Sometimes it is easier to argue by contradiction, i.e. to assume that the desired conclusion is false and derive a contradiction to some known fact.

Counter-examples

To show that a statement is false it is enough to give a single instance for which it does not hold, called a counter-example.

METHODS OF PROOF – MATHEMATICAL INDUCTION

The term induction involves deriving of a general statement or rule from one or more particular cases. A statement may be true for a few special cases. Does it follow, then, that it is true in all cases?

Consider the following two examples:

1. It is true that 60 is divisible by 1, 2, 3, 4, 5 and 6. This establishes a pattern. Does it follow that 60 is divisible by all positive integers?

$$1 = 1^2$$

$$1 + 3 = 2^2$$

$$1 + 3 + 5 = 3^2$$

$$1 + 3 + 5 + 7 = 4^2$$

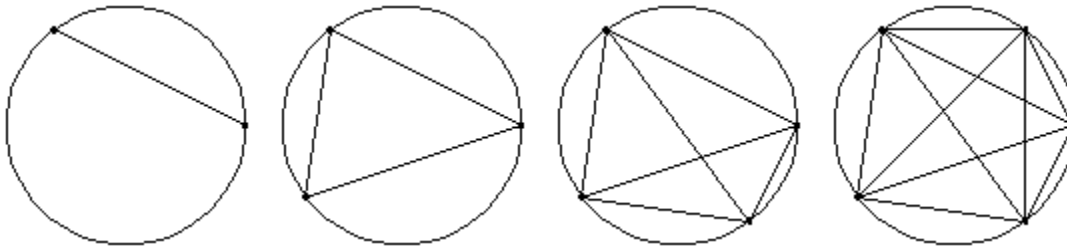
2. The sum of the first two odd numbers is a perfect square, so too is the sum of the first three odd numbers, and of the first five odd numbers. Does it follow that the sum of the first n numbers is a perfect square? The pattern seems to indicate that it is. We shall proof shortly that it is.

The need for proof

If we see something that works a few times in a row, we're convinced that it works forever.

Regions of a Circle

Consider a circle with n points on it. How many regions will the circle be divided into if each pair of points is connected with a chord?



2 points
2 regions = 2^1

3 points
4 regions = 2^2

4 points
8 regions = 2^3

5 points
16 regions = 2^4

At this point, probably everyone would be convinced that with 6 points there would be 32 regions, but it's not proved, it's just conjectured. The conjecture is that the number of regions when n points are connected is 2^{n-1} .

Will finding the number of regions when there are six points on the circle prove the conjecture? No. If there are indeed 32 regions, all you have done is shown another example to support your conjecture. If there aren't 32 regions, then you have proved the conjecture wrong. In fact, if you go ahead and try the circle with six points on it, you'll find out that there aren't 32 regions.

You can never prove a conjecture is true by example.

You can prove a conjecture is false by finding a counter-example.

To prove a conjecture is true, you need some more formal methods of proof. One of these methods is the principle of mathematical induction.

Proof by induction

The method of proof by induction is applied to propositions, $P(n)$, containing natural numbers n .

Step 1: Verify that the statement is true for a specific case, usually $n = 1 \rightarrow P(1)$ is true.

Step 2: Assume that the statement is true for some integer, say $n = k$, and then prove that it must be true for $n = k + 1$. In other words prove $P(k) \rightarrow P(k + 1)$.

Step 3: If it is true for $n = 1$, then it must be true for $n = 2$, and then being true for

$n = 2$, it must be true for $n = 3$ and so on. So it is true for all natural numbers. $P(n)$ is true for every n .

Example 1

Prove, by mathematical induction, that the sum of the first n odd integers is n^2 .

$$1 + 3 + 5 + 7 + \dots + (2n - 1) = n^2$$

Solution

Step 1:

When $n = 1$

$$LHS = 1$$

$$RHS = 1^2 = 1$$

$$\therefore LHS = RHS$$

And the statement is true for $n = 1$.

Step 2:

Assume the statement is true for $n = k$.

$$1 + 3 + 5 + 7 + \dots + (2k - 1) = k^2$$

Step 3

Based on the assumption in Step 2, show that the following is true, by replacing k with $k+1$

$$1 + 3 + 5 + 7 + \dots + (2k - 1) + [2(k + 1) - 1] = (k + 1)^2$$

We proceed as follows:

$$\underbrace{1 + 3 + 5 + 7 + \dots + (2k - 1)}_{=k^2 \text{ (from Step 2)}} + [2(k + 1) - 1] = (k + 1)^2$$

$$LHS = k^2 + 2k + 2 - 1 = k^2 + 2k + 1 = (k + 1)^2 = RHS$$

Conclusion:

We showed it is true for $n = 1$ and we showed that it follows $n = k \text{ true} \Rightarrow n = k + 1 \text{ true}$, so the statement is true for any positive integer by the principle of mathematical induction.

Principle of Mathematical Induction (common language)

1. Show something works the first time.
2. Assume that it works for this time,
3. Show it will work for the next time.
4. Conclusion, it works all the time

Principle of Mathematical Induction (mathematical language)

To prove a particular proposition $P(n)$ for $n \in \mathbb{Z}^+$.

1. Show true for $n = 1$. $P(1)$ true.
2. Assume true for $n = k$. $P(k)$ true
3. Show true for $n = k + 1$ given true for $n = k$. $P(k)$ true $\Rightarrow P(k + 1)$ true
4. Conclusion: Statement is true for all $n \geq 1$
5. Write the final statement: Since $P(1)$ is true and $P(k)$ true implies $P(k + 1)$ true, therefore the statement is true by the principle of mathematical induction.

The key word in step 2 is **assume**. You are not trying to prove it's true for $n = k$, you're going to accept on faith that it is, and show it's true for the next number, $n = k + 1$. If it later turns out that you get a contradiction, then the assumption was wrong.

The **Principle of Mathematical Induction** can be informally stated by saying:

"We can establish the truth of a proposition if we can show that it follows from smaller instances of the same proposition, as long as we can establish the truth of the smallest instance (or instances) explicitly." (Grossman)

In induction, **the truth percolates up through the layers to prove the whole proposition**. If we know the first instance of a proposition is true and the second one is derived from the 1st one, making it true, and the pattern repeats itself, then we know that our final result will also be true. But if the first, or any of the other instances following the first, turn out to be false then the final result will be wrong.

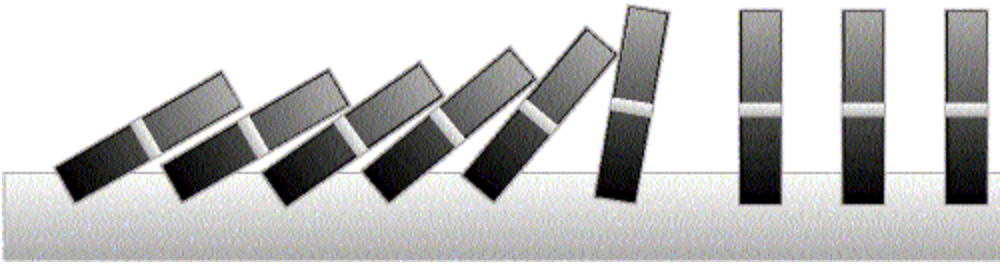
Falling Dominoes

<http://www.youtube.com/watch?v=W9ZyxFQjeg0>

Let's look at a row of dominoes, lined up and ready to be pushed over.

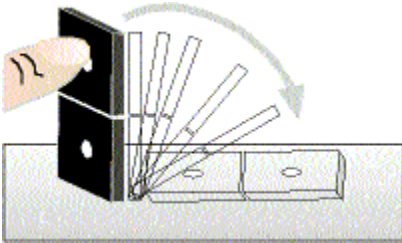


We know from experience that if we push over one domino, the rest of the dominoes will fall over.

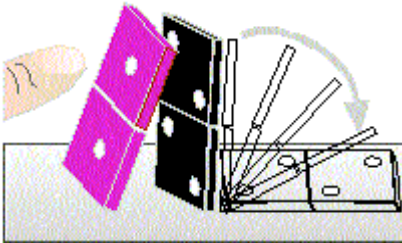


How can we prove this?

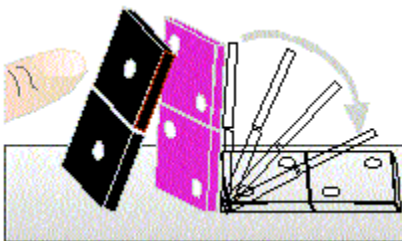
A. We know from experience that if we push over one domino, it should fall over.



B. We also know that if a domino is falling and has been placed correctly, it will knock over its neighbour.



Intuitively, it should be very clear that the fall should cascade all the way up to the last domino. That is, if the next to the last domino falls, the last domino also falls.



But now we need to think about increasing the rigor of this argument by doing a real proof.




A Formal Proof

Let's look at the behaviour of the dominoes.

1. Assume that there's some domino k which doesn't fall over. i.e. k is the first domino which doesn't work right (maybe it's out of line, too far away, or has a different gravitational constant).
2. Since k is the first such domino, the domino right before k must have fallen over.





3. But we know from **B** that a falling domino always knocks its neighbour over.
4. So domino k will fall over and we have a contradiction.

What we have shown above is that because

1. **We can knock over the first domino.** 
2. **and a falling domino knocks over its neighbour.** 
3. **then *all the dominoes will fall over.*** 

Now, if we think of each domino as an instance of a proposition. If a given instance is true, the corresponding domino will fall over. Given a sequence of instances (row of dominoes):

If we can prove:

1. The proposition is true in the first instance. 
2. And if a given instance is true,  the next one in the sequence will also be true. 
3. Then the proposition will be true in all instances. 

This is called a **Proof by Induction**.

Two more examples are shown in the following Power Point:

Mathematical induction applies to propositions involving divisibility and to propositions involving matrix equations. It is illustrated in the examples below.

Example.

Prove that $2^{4n+1} + 3$ is a multiple of 5 for any positive integer n .

Solution

Let $f(n) = 2^{4n+1} + 3$.

Step 1: Prove $f(1)$ is a multiple of 5. But $f(1) = 2^5 + 3 = 35$ (which is a multiple of 5).

Step 2: We assume $f(k)$ is a multiple of 5. We want to prove that $f(k + 1)$ is a multiple of 5.

First we try to simplify:

$$f(n + 1) - f(n) = 2^{4(n+1)+1} + 3 - (2^{4n+1} + 3) = 2^{4n+5} - 2^{4n+1} = 2^{4n+1}(2^4 - 1) = 15 \times 2^{4n+1}$$

Therefore:

$$f(k + 1) - f(k) = 15 \times 2^{4k+1}$$

$$\text{So, } f(k + 1) = 15 \times 2^{4k+1} + f(k).$$



Multiple of 5

Assumed to be a multiple of 5

So, $f(k + 1)$ is a multiple of 5.

Therefore by induction, $f(n)$ must be a multiple of 5 for all positive integers n .

TRY THIS

Prove that $2^{3n+2} + 5^{n+1}$ is divisible by 3 for any positive integer n .

ANSWER

Let $f(n) = 2^{3n+2} + 5^{n+1}$

If we here calculate, $f(n + 1) + f(n) = 2^{3(n+1)+2} + 5^{n+1+1} + (2^{3n+2} + 5^{n+1})$

$$= 2^{3n+5} + 5^{n+2} + 2^{3n+2} + 5^{n+1}$$

$$= 2^{3n+5} + 2^{3n+2} + 5^{n+2} + 5^{n+1}$$

$$= 2^{3n+2}(2^3 + 1) + 5^{n+1}(5 + 1)$$

$$= 9 \times 2^{3n+2} + 6 \times 5^{n+1}$$

Now we are ready to prove the result.

Step 1: First prove true when $n = 1$:

$$2^{3+2} + 5^{1+1} = 32 + 25 = 57 \quad (\text{which is divisible by } 3).$$

Step 2: Assume that the result is true when $n = k$, i.e. that $f(k)$ is divisible by 3.

We showed above that $f(k + 1) + f(k) = 9 \times 2^{3k+2} + 6 \times 5^{k+1}$

i.e. that

$$f(k + 1) = 9 \times 2^{3k+2} + 6 \times 5^{k+1} - f(k)$$

Divisible by 3

Divisible by 3

Divisible by 3
(by assumption)

So $f(k + 1)$ must be divisible by 3 (as required).

Therefore the result is true for $n \in \mathbb{Z}^+$.

Example

Prove that for all natural numbers $\frac{1}{2} + \frac{1}{6} + \frac{1}{12} + \dots + \frac{1}{n(n+1)} = \frac{n}{n+1}$.

Solution

Step 1

Is this true for $n = 1$?

$$\frac{1}{2} = \frac{1}{1+1}$$

$$\frac{1}{2} = \frac{1}{2}$$

True for $n = 1$.

Step 2

Assume it is true for $n = k$:

$$\frac{1}{2} + \frac{1}{6} + \frac{1}{12} + \dots + \frac{1}{k(k+1)} = \frac{k}{k+1}.$$

Step 3

Based on the assumption in Step 2, show that the following is true, by replacing k with $k+1$

$$\underbrace{\frac{1}{2} + \frac{1}{6} + \frac{1}{12} + \dots + \frac{1}{k(k+1)}}_{\frac{k}{k+1}} + \frac{1}{(k+1)(k+1+1)} = \frac{k+1}{k+1+1}$$

$$\frac{k}{k+1} + \frac{1}{(k+1)(k+2)} = \frac{k+1}{k+2}$$

Find the common denominator on the LHS:

$$\begin{aligned} LHS &= \frac{k(k+2)}{(k+1)(k+2)} + \frac{1}{(k+1)(k+2)} \\ &= \frac{k^2 + 2k + 1}{(k+1)(k+2)} \end{aligned}$$

Factorize the denominator now:

$$LHS = \frac{(k+1)(k+1)}{(k+1)(k+2)} = \frac{k+1}{k+2} = RHS$$

LHS = RHS, so the result is true for $n = k + 1$.

True for $n = 1$ and $P(k)$ true $\Rightarrow P(k+1)$ true, therefore true for all positive integer values of n .

Example

Using mathematical induction, prove that $\frac{d^n}{dx^n}(\cos x) = \cos\left(x + \frac{n\pi}{2}\right)$, for all positive integer values of n .

Solution

Let p_n be the statement $\frac{d^n}{dx^n} \cos x = \cos\left(x + \frac{n\pi}{2}\right)$ for all positive integer values of n .

$$\text{For } n = 1, \frac{d}{dx}(\cos x) = -\sin x = \cos\left(x + \frac{\pi}{2}\right)$$

Therefore p_1 is true.

Assume the formula is true for $n = k$, that is, $\frac{d^k}{dx^k}(\cos x) = \cos\left(x + \frac{k\pi}{2}\right)$

$$\text{Then } \frac{d}{dx} \left(\frac{d^k}{dx^k} (\cos x) \right) = \frac{d}{dx} \left(\cos \left(x + \frac{k\pi}{2} \right) \right)$$

$$\frac{d^{k+1}}{dx^{k+1}} (\cos x) = -\sin \left(x + \frac{k\pi}{2} \right)$$

$$\frac{d^{k+1}}{dx^{k+1}} (\cos x) = \cos \left(x + \frac{k\pi}{2} + \frac{\pi}{2} \right)$$

$$\frac{d^{k+1}}{dx^{k+1}} (\cos x) = \cos \left(x + \frac{(k+1)\pi}{2} \right)$$

which is p_n when $n = k + 1$.

So if p_n is true for $n = k$ then it is true for $n = k + 1$ and by the principle of mathematical induction p_n is true for all positive integer values of n .

Lesson Summary

Mathematical induction is a method of proving statements involving the natural numbers 1, 2, 3, . . .

The idea is that

(i) we prove the statement when $n = 1$

and (ii) show that if the statement is true for some integer k then it is true for the integer immediately above.

From this we can conclude that the statement is true for all $n = 1, 2, 3, \dots$

The Principle of Mathematical Induction.

Let $P(n)$ be a statement depending on an arbitrary positive integer n . Suppose that we can do the following two steps:

(i) Verify that $P(1)$ is true,

(ii) Assume that $P(k)$ is true and show that if $P(k)$ is true then $P(k + 1)$ is true.

Then the statement $P(n)$ is true for all positive integers n .

The statement $P(n)$ is called the inductive hypothesis, step (i) is called starting the induction and step (ii) is called the inductive step.

Note that the Principle of Induction is intuitively obvious: if $P(1)$ is true

and $P(k) \Rightarrow P(k+1)$, that is the truth of $P(k)$ implies the truth of $P(k+1)$ for all $n = 1, 2, 3, \dots$, then

$P(1) \Rightarrow P(2) \Rightarrow P(3) \Rightarrow \dots \Rightarrow P(n) \dots$

by applying (ii) with $n = 1, 2, 3, \dots$ in turn, so $P(n)$ is true for all n .